Question 1 *(12 Marks)* **Marks**

(a) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$. **2**

(b) The point P(x, y) divides the interval AB externally in the ratio 4:3 where A is the point (2,–1) and B is the point (1,–3). Find the value of x and y. **2**

(c) State the domain and range of $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$. **2**

(d) The polynomial $P(x) = x^3 + ax^2 - 3x + 5$ has a remainder of 6 when divided by $(x+1)$. Find the values of a. **2**

(e) Find the acute angle between the lines $2x+3y+5=0$ and $3x-4y=12$. Answer correct to the nearest minute. **2**

(f) Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{4x}$. **2**

Question 2 *(12 Marks)* Start a new booklet.

(a) Use the substitution $u = x-2$ to evaluate $\displaystyle\int_4^5 \frac{x(x-4)}{(x-2)}dx$. **4**

(b) The equation $x^3 - 5 = 0$ is used to approximate the cubed root of 5. Given that the root of the equation is near 2, use two approximations of Newton's method to find an approximate root, correct to 3 decimal places. **3**

(c) Solve $\dfrac{2x}{x+1} \le x$. **3**

(d) Find the coefficient of x^6 in the expansion $\left(x^2+4\right)^{12}$. **2**

Marks

Question 3 *(12 Marks)* Start a new booklet.

(a) Use mathematical induction to prove that for all positive integers, n **4**

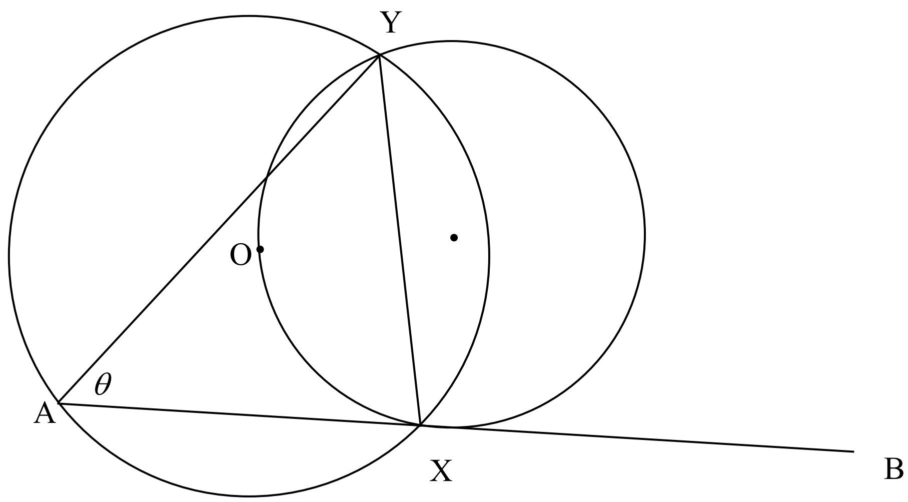
$$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2.$$

(b) From a point A, 200m due south of a cliff, the angle of elevation of the top of the cliff is 30°. From a point B, due east of the cliff, the angle of elevation of the top of the cliff is 20°

(i) Draw a diagram showing all this information. **1**

(ii) Find the distance between A and B. Answer correct to the nearest metre. **3**

(c) O is the centre of the larger circle. The two circles intersect at the points X and Y. AXB is a tangent to the smaller circle at point X. O is on the circumference of the smaller circle.



Copy or trace the diagram onto your answer paper.

(i) Find $\angle XOY$ in terms of θ. Give a reason for your answer. **1**

(ii) Explain why $\angle BXY = 2\theta$. **1**

(iii) Prove AX=YX. **2**

Question 4 *(12 Marks)* Start a new booklet. **Marks**

(a) The points P $(2ap, ap^2)$ and $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the vertex of the parabola. The normals at P and Q meet at T.

(i) Show that $pq = -4$. **2**

(ii) Show that the equation of the normal at P is $x + py = ap^3 + 2ap$ **2**

(iii) Show that T has the coordinates **3**

$(-apq(p+q), a(p^2 + pq + q^2 + 2))$.

(iv) Find the Cartesian equation of the locus of T. **3**

(b) Prove $\dfrac{\sin 5\theta}{\sin\theta} - \dfrac{\cos 5\theta}{\cos\theta} = 4\cos 2\theta$ **2**

Question 5 *(12 Marks)* Start a new booklet.

(a) A particle is moving in Simple Harmonic Motion about the origin. **2**

(i) Assuming that $v^2 = n^2(a^2 - x^2)$, show that $\ddot{x} = -n^2 x$ where a is the amplitude.

(ii) When the particle is 4 metres from the origin its speed is $6ms^{-1}$ and when it is 3 metres from the origin its speed is $8ms^{-1}$. **3**
Find the amplitude and period of the motion.

(iii) Find the greatest acceleration of the particle. **1**

(b) The annual growth rate of the population of a NSW country town is projected to be 15% of the excess of the population that is over 30000. Initially in 2008, the population was 32000.

(i) Show that $P = 30000 + Ae^{0.15t}$ is a solution to the differential equation $\dfrac{dP}{dt} = 0.15(P - 30000)$ and hence find A. **2**

(ii) Determine the population after 10 years. **2**

(iii) Determine how long it will take for the population to reach 50000. **2**

Question 6 *(12 Marks)* Start a new booklet. **Marks**

(a) The velocity of a particle is given by $v = 3x + 7$. If the initial displacement is $1cm$ to the right of the origin, find the displacement after 5 seconds. **3**

(b) Solve $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x+1)$ **3**

(c) In how many ways can the letters of the word PROBABILITY be arranged in a circle? **2**

(d) (i) By considering the terms in x^r on both sides of the identity

$(1+x)^{m+n} = (1+x)^m (1+x)^n$, show that ${}^{m+n}C_r = \sum_{k=0}^{r} {}^mC_k \, {}^nC_{r-k}$ **2**

for $0 \le r \le m$ and $0 \le r \le n$.

(ii) Hence show that **2**

$${}^{m+1}C_0 \, {}^nC_2 + {}^{m+1}C_1 \, {}^nC_1 + {}^{m+1}C_2 \, {}^nC_0 = {}^mC_0 \, {}^{n+1}C_2 + {}^mC_1 \, {}^{n+1}C_1 + {}^mC_2 \, {}^{n+1}C_0$$

for $m \ge 2$ and $n \ge 2$.

Question 7 *(12 Marks)* Start a new booklet.

(a) Show that $\frac{d}{dx}\left(x \tan^{-1} x\right) = \frac{x}{1+x^2} + \tan^{-1} x$. Hence evaluate $\int_0^1 \tan^{-1} x\,dx$. **4**

(b) A baseball player hits the ball from ground level with a speed of $20ms^{-1}$ and an angle of elevation, α. It flies towards a building 20 metres away on level ground. Given the equations of motion are

$$x = 20t\cos\alpha$$
$$y = -5t^2 + 20t\sin\alpha$$

(i) Find the Cartesian equation of the path of the ball in flight. **1**

(ii) Show that the height h at which the ball hits the wall is given by **1**
$h = 20\tan\alpha - 5(1+\tan^2\alpha)$.

(iii) Using part (ii) above, show that the maximum value of h occurs when $\tan\alpha = 2$. **2**

(iv) Find the maximum height. **2**

(v) Find the speed at which the ball hits the wall. **2**

End of paper

Year 12, 2009 Extension 1 Mathematics Trial Examination

Question 1

a) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}0$

$= \frac{\pi}{3}$

b) $A(2,-1)$ $B(1,-3)$ $m:n = 4:-3$

$x = \frac{nx_1+mx_2}{m+n}$ $\quad$ $y = \frac{ny_1+my_2}{m+n}$

$= \frac{-3(2)+4(1)}{4-3}$ $\quad$ $= \frac{-3(-1)+4(-3)}{4-3}$

$= -2$ $\quad$ $= 9$

$\therefore$ $(-2,9)$ divides the interval AB externally in the ratio 4:3

c) $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$

Domain: $-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$

Range: $0 \le \cos^{-1}\left(\frac{x}{3}\right) \le \pi$

$0 \le 2\cos^{-1}\left(\frac{x}{3}\right) \le 2\pi$

d) $P(x) = x^3 + ax^2 - 3x + 5$

$P(-1) = (-1)^3 + a(-1)^2 - 3(-1) + 5 = 6$

$-1 + a + 3 + 5 = 6$

$a = -1$

e) $2x + 3y + 5 = 0$

$3y = -2x - 5$

$y = -\frac{2}{3}x - \frac{5}{3}$

$m_1 = -\frac{2}{3}$

$3x - 4y = 12$

$-4y = -3x - 12$

$y = \frac{3}{4}x + 3$

$m_2 = \frac{3}{4}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-\frac{2}{3} - \frac{3}{4}}{1 + \frac{3}{4} \times -\frac{2}{3}}\right|$

$= \frac{17}{6}$

$\theta = 70°34'$ to nearest minute

f) $\lim_{x\to 0} \frac{\sin 2x}{4x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin 2x}{2x}$

$= \frac{1}{2}$

Question 2

a) $\int_4^5 \frac{x(x-4)}{(x-2)}\,dx$

$u = x - 2$

$x = u + 2$

$du = dx$

when $x = 5$, $u = 3$

when $x = 4$, $u = 2$

$= \int_2^3 \frac{(u+2)(u-2)}{u}\,du$

$= \int \frac{u^2-4}{u}\,du$

$= \int\left(u - \frac{4}{u}\right)du$

$= \left[\frac{1}{2}u^2 - 4\ln u\right]_2^3$

$= \left[\frac{1}{2}(3)^2 - 4\ln 3\right] - \left[\frac{1}{2}(2)^2 - 4\ln 2\right]$

$= \frac{9}{2} - 4\ln 3 - 2 + 4\ln 2$

$= \frac{5}{2} + 4\ln\left(\frac{2}{3}\right)$

b) Let $f(x) = x^3 - 5$

$f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{3}{12}$$

$$= 1.75$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.75 - \frac{f(1.75)}{f'(1.75)}$$

$$\doteqdot 1.710884354$$

$$= 1.711 \text{ to 3dp}$$

$\therefore$ An approximate root is 1.711 to 3dp.

c) $\dfrac{2x}{x+1} \leq x \qquad \boxed{x \neq -1}$

$$2x(x+1) \leq x(x+1)^2$$

$$x(x+1)^2 - 2x(x+1) \geq 0$$

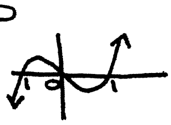
$$(x+1)[x(x+1) - 2x] \geq 0$$

$$(x+1)(x^2 + x - 2x) \geq 0$$

$$(x+1)(x^2 - x) \geq 0$$

$$(x+1)x(x-1) \geq 0$$

$\therefore -1 < x \leq 0,\ x \geq 1$



d) $(x^2+4)^{12}$

General Term $= {}^{12}C_r (x^2)^{12-r} (4)^r$

$$= {}^{12}C_r\, x^{24-2r}\, 4^r$$

To obtain the term in x^6

$$24 - 2r = 6$$

$$-2r = -18$$

$$r = 9$$

$\therefore$ Coefficient of $x^6 = {}^{12}C_9\, 4^9$

$$= 57671680$$

Question 3

a) $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$

ie. $1^3 + 2^3 + 3^3 + \dots + n^3 = \dfrac{n^2}{4}(n+1)^2$

Step 1 Test for $n=1$

$$L.HS = 1^3$$

$$= 1$$

$$RHS = \frac{1^2}{4}(1+1)^2$$

$$= 1$$

$$L.HS = RHS$$

$\therefore$ Result is true for $n=1$

Step 2 Assume the result is true for $n=k$, that is assume

$$S_k = \frac{k^2}{4}(k+1)^2$$

Step 3 Hence show the result is true for $n=k+1$ that is show

$$S_{k+1} = \frac{(k+1)^2}{4}(k+2)^2$$

$$S_{k+1} = S_k + T_{k+1}$$

$$= \frac{k^2}{4}(k+1)^2 + (k+1)^3$$

$$= (k+1)^2\left[\frac{k^2}{4} + (k+1)\right]$$

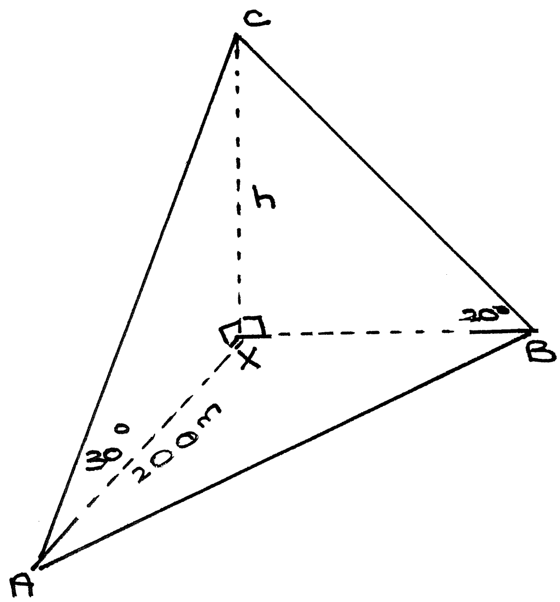
$$= \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right]$$

$$= \frac{(k+1)^2}{4}(k+2)^2$$

Hence if the result is true for $n=k$ then it is true for $n=k+1$

Step 4 Since the result is true for $n=1$ then from Step 3 it is true for $n=1+1=2$ and then for $n=3$ and so on for all positive integral values of n.

b)



Let h be the height of the cliff

In $\triangle CXA$, $\frac{h}{200} = \tan 30°$

$h = 200 \tan 30°$ ①

In $\triangle CXB$ $\frac{h}{XB} = \tan 20$

$XB = \frac{h}{\tan 20}$ ②

sub ① in ②

$XB = \frac{200 \tan 30}{\tan 20}$

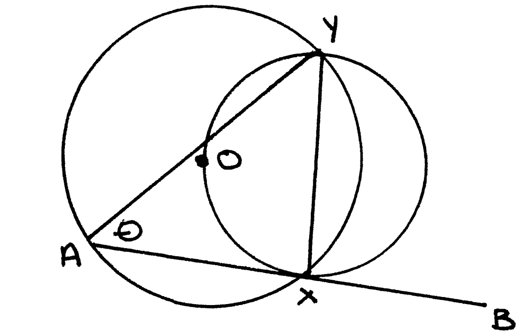
$\therefore AB^2 = AX^2 + XB^2$

$= 200^2 + \left(\frac{200 \tan 30}{\tan 20}\right)^2$

$\doteq 140648.4289$

$AB = 375.0312373$

$\therefore$ The distance between A and B is 375 m to the nearest metre.



(i) $X\hat{O}Y = 2\theta$ Angle at the centre is twice the angle at the circumference standing on the same arc

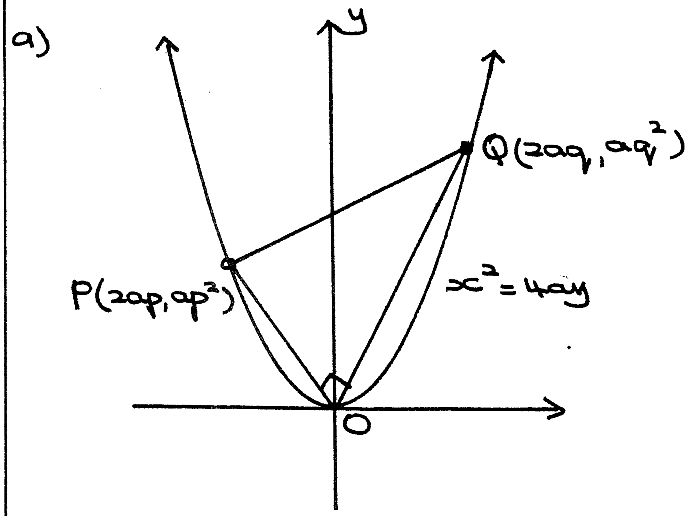
(ii) $B\hat{X}Y = X\hat{O}Y$
$= 2\theta$ The angle between a tangent to a circle and a chord at the point of contact is equal to any angle in the alternate segment

$B\hat{X}Y = X\hat{A}Y + A\hat{Y}X$ Exterior angle of a triangle is equal to the sum of the 2 interior opposite angles
$2\theta = \theta + A\hat{Y}X$
$A\hat{Y}X = \theta$

$\therefore A\hat{Y}X = X\hat{A}Y = \theta$ In an isosceles triangle equal angles are opposite equal sides.
$\therefore AX = YX$

Question 4

a)



i) $m_{PO} = \dfrac{ap^2 - 0}{2ap - 0}$

$= \dfrac{p}{2}$

$m_{QO} = \dfrac{aq^2 - 0}{2aq - 0}$

$= \dfrac{q}{2}$

For perpendicular lines $m_1 m_2 = -1$

$\therefore \dfrac{p}{2} \times \dfrac{q}{2} = -1$

$pq = -4$

ii) $x^2 = 4ay$

$y = \dfrac{x^2}{4a}$

$\dfrac{dy}{dx} = \dfrac{2x}{4a}$

$= \dfrac{x}{2a}$ at $x = 2ap$

$= \dfrac{2ap}{2a}$

$= p$

Gradient of tangent at $P = p$

Gradient of normal at $P = -\dfrac{1}{p}$

(as $m_1 m_2 = -1$ for perpendicular lines)

$y - y_1 = m(x - x_1)$

$y - ap^2 = -\dfrac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

(iii) Equation of the normal at P

$x + py = 2ap + ap^3$ —— ①

similarly, the equation of the normal at Q is

$x + qy = 2aq + aq^3$ —— ②

① − ② $\quad py - qy = 2a(p - q) + a(p^3 - q^3)$

$y(p - q) = 2a(p - q) + a(p - q)(p^2 + pq + q^2)$

$y = \dfrac{a(p - q)(p^2 + pq + q^2 + 2)}{(p - q)}$

$= a(p^2 + pq + q^2 + 2)$

sub ①

$x + py = 2ap + ap^3$

$x + ap(p^2 + pq + q^2 + 2) = 2ap + ap^3$

$x + \cancel{ap^3} + ap^2q + apq^2 + \cancel{2ap} = \cancel{2ap} + \cancel{ap^3}$

$\therefore x = -ap^2q - apq^2$

$= -apq(p + q)$

$\therefore$ T is $\left(-apq(p + q),\ a(p^2 + pq + q^2 + 2)\right)$

(iv) Using part (i) $\Rightarrow pq = -4$

T is $\left(4a(p + q),\ a(p^2 - 4 + q^2 + 2)\right)$

$\therefore x = 4a(p + q)$

$p + q = \dfrac{x}{4a}$ —— ①

$y = a(p^2 - 4 + q^2 + 2)$

$= a(p^2 + q^2 - 2)$

$= a\left[(p + q)^2 - 2pq - 2\right]$

$$y = a\left[(p+q)^2 + 8 - 2\right] \quad \text{as } pq = -4$$

$$= a\left[(p+q)^2 + 6\right]$$

$$= a\left[\left(\frac{x}{4a}\right)^2 + 6\right] \quad \text{sub } ①$$

$$= a\left(\frac{x^2}{16a^2} + 6\right)$$

$$= a\left(\frac{x^2 + 96a^2}{16a^2}\right)$$

$$16ay = x^2 + 96a^2$$

$$x^2 = 16ay - 96a^2$$

$$x^2 = 16a(y-6)$$

b) $\dfrac{\sin 5\theta}{\sin\theta} - \dfrac{\cos 5\theta}{\cos\theta} = 4\cos 2\theta$

$$\text{L.HS} = \frac{\sin 5\theta}{\sin\theta} - \frac{\cos 5\theta}{\cos\theta}$$

$$= \frac{\sin 5\theta \cos\theta - \cos 5\theta \sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(5\theta - \theta)}{\frac{1}{2}\sin 2\theta}$$

$$= \frac{\sin 4\theta}{\frac{1}{2}\sin 2\theta}$$

$$= \frac{2\sin 2\theta\cos 2\theta}{\frac{1}{2}\sin 2\theta}$$

$$= 4\cos 2\theta$$

$$= \text{RHS}$$

$$\therefore \frac{\sin 5\theta}{\sin\theta} - \frac{\cos 5\theta}{\cos\theta} = 4\cos 2\theta$$

Question 5

a) i) $v^2 = n^2(a^2 - x^2)$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}n^2(a^2 - x^2)\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}n^2a^2 - \frac{1}{2}n^2x^2\right)$$

$$= -n^2x$$

$$\therefore \ddot{x} = -n^2x$$

ii) $v^2 = n^2(a^2 - x^2)$

when $x = 4$, $v = 6$

$$36 = n^2(a^2 - 16) \quad ①$$

when $x = 3$, $v = 8$

$$64 = n^2(a^2 - 9) \quad ②$$

$$② \div ① \quad \frac{64}{36} = \frac{a^2 - 9}{a^2 - 16}$$

$$\frac{16}{9} = \frac{a^2 - 9}{a^2 - 16}$$

$$9a^2 - 81 = 16a^2 - 256$$

$$7a^2 = 175$$

$$a^2 = 25$$

$$a = 5 \quad a > 0$$

sub ① $36 = n^2(a^2 - 16)$

$$36 = n^2(25 - 16)$$

$$36 = 9n^2$$

$$n = 2 \quad n > 0$$

$$T = \frac{2\pi}{n}$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

$\therefore$ amplitude is 5 metres

period is π seconds

(iii) Maximum acceleration occurs when $x = a$

$$\ddot{x} = -n^2x \quad n = 2,\ x = 5$$

$$= -20$$

maximum acceleration is $20\,ms^{-2}$

(negative implies direction

b) i) $P = 30000 + Ae^{0.15t}$ — (1)

$$\frac{dP}{dt} = 0.15Ae^{0.15t}$$
$$= 0.15(P - 30000)$$

from (1) $Ae^{0.15t} = P - 30000$

when $t = 0$, $P = 32000$

$$P = 30000 + Ae^{0.15t}$$
$$32000 = 30000 + Ae^{0.15(0)}$$
$$2000 = A$$

ii) $\therefore P = 30000 + 2000e^{0.15t}$
$$= 30000 + 2000e^{0.15 \times 10}$$
$$\doteqdot 38963.37814$$

$\therefore$ Population is approximately 38963

iii) $P = 30000 + 2000e^{0.15t}$
$$50000 = 30000 + 2000e^{0.15t}$$
$$20000 = 2000e^{0.15t}$$
$$10 = e^{0.15t}$$
$$0.15t = \ln 10$$
$$t = \frac{\ln 10}{0.15}$$
$$\doteqdot 15.35056729$$

$\therefore$ It takes approximately 15.35 years to reach 50000

Question 6

a) $v = 3x + 7$

$$\frac{dx}{dt} = 3x + 7$$

$$\frac{dt}{dx} = \frac{1}{3x+7}$$

$$t = \int \frac{1}{3x+7}\,dx$$

$$t = \ln(3x+7) + c$$

when $t = 0$ $x = 1$

$$\therefore 0 = \ln 10 + c$$
$$c = -\ln 10$$
$$\therefore t = \ln(3x+7) - \ln 10$$
$$= \ln\left(\frac{3x+7}{10}\right)$$

$$e^t = \frac{3x+7}{10}$$

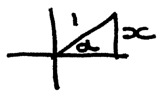
$$3x + 7 = 10e^t$$
$$3x = 10e^t - 7$$
$$x = \tfrac{1}{3}(10e^t - 7)$$

b) $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x+1)$

Let $\alpha = \sin^{-1}x$

$\sin\alpha = x$

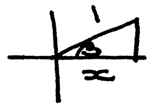
$\cos\alpha = \sqrt{1-x^2}$



Let $\beta = \cos^{-1}x$

$\cos\beta = x$

$\sin\beta = \sqrt{1-x^2}$



$$\sin(\alpha - \beta) = \sin(\sin^{-1}(3x+1))$$
$$x^2 - \sqrt{1-x^2}\sqrt{1-x^2} = 3x+1$$
$$x^2 - (1-x^2) = 3x+1$$
$$2x^2 - 1 = 3x+1$$
$$2x^2 - 3x - 2 = 0$$
$$(2x+1)(x-2) = 0$$
$$x = -\tfrac{1}{2}, 2 \quad (-1 \le x \le 1)$$

$\therefore x = -\frac{1}{2}$ is the only solution

c) Number of ways $= \frac{11!}{2!2!}$
$$= 9979200$$

i) $(1+x)^{m+n} = (1+x)^m (1+x^n)$

L.HS $= (1+x)^{m+n}$

$= 1 + {}^{m+n}C_1 x + {}^{m+n}C_2 x^2 + {}^{m+n}C_3 x^3 + \dots {}^{m+n}C_r x^r + \dots + {}^{m+n}C_{m+n} x^{m+n}$

R.HS $= (1+x)^m (1+x^n)$

$= (1 + {}^mC_1 x + {}^mC_2 x^2 + {}^mC_3 x^3 + \dots + {}^mC_r x^r + \dots + {}^mC_m x^m) \times$

$(1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n)$

The coefficient of x^r on LHS $= {}^{m+n}C_r$

The coefficient of x^r on RHS $= \sum_{k=0}^{r} {}^mC_k \, {}^nC_{r-k}$

Equating coefficients ${}^{m+n}C_r = \sum_{k=0}^{r} {}^mC_k \, {}^nC_{r-k}$

ii) Using (i) $m \geq 2, n \geq 2$

$${}^{m+1}C_0 \, {}^nC_2 + {}^{m+1}C_1 \, {}^nC_1 + {}^{m+1}C_2 \, {}^nC_0 = {}^{(m+1)+n}C_2$$

$${}^mC_0 \, {}^{n+1}C_2 + {}^mC_1 \, {}^{n+1}C_1 + {}^mC_2 \, {}^{n+1}C_0 = {}^{m+(n+1)}C_2$$

$$\therefore {}^{m+1}C_0 \, {}^nC_2 + {}^{m+1}C_1 \, {}^nC_1 + {}^{m+1}C_2 \, {}^nC_0 = {}^mC_0 \, {}^{n+1}C_2 + {}^mC_1 \, {}^{n+1}C_1 + {}^mC_2 \, {}^{n+1}C_0$$

<u>Question 7</u>

a) Let $y = x\tan^{-1}x$

$= uv$

$u = x$ $\quad v = \tan^{-1}x$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = \frac{1}{1+x^2}$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$= \tan^{-1}x + \frac{x}{1+x^2}$$

$$\therefore \frac{d}{dx}(x\tan^{-1}x) = \frac{x}{1+x^2} + \tan^{-1}x$$

$$\int \left(\frac{x}{1+x^2} + \tan^{-1}x\right) dx = x\tan^{-1}x$$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$$

$$\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$\int_0^1 \tan^{-1}x\,dx = \left[1\tan^{-1}1\right] - 0 - \left[\tfrac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2 - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

b) i) $x = 20t\cos\alpha$ —— ①

$y = -5t^2 + 20t\sin\alpha$ — ②

Rearrange ①

$$t = \frac{x}{20\cos\alpha}$$

Sub ②

$$y = -5t^2 + 20t\sin\alpha$$

$$= -5\left(\frac{x}{20\cos\alpha}\right)^2 + 20\left(\frac{x}{20\cos\alpha}\right)\sin\alpha$$

$$\therefore y = -\frac{x^2}{80}\sec^2\alpha + x\tan\alpha$$

ii) when $x = 20$, $y = h$

$$y = -\frac{x^2}{80}\sec^2\alpha + x\tan\alpha$$

$$h = -\frac{400}{80}\sec^2\alpha + 20\tan\alpha$$

$$h = 20\tan\alpha - 5\sec^2\alpha$$

$$= 20\tan\alpha - 5(1+\tan^2\alpha)$$

iii) $\frac{dh}{d\alpha} = 20\sec^2\alpha - 10\tan\alpha\sec^2\alpha$

For stationary points $\frac{dh}{dt} = 0$

$$20\sec^2\alpha - 10\tan\alpha\sec^2\alpha = 0$$

$$\sec^2\alpha(2 - \tan\alpha) = 0$$

$$\sec\alpha = 0 \text{ or } \tan\alpha = 2$$

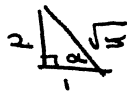
$$\therefore \tan\alpha = 2$$

Test the nature

α	63^{0-}	63^0	63^{0+}
$\frac{dh}{dt}$	+		−

$\therefore$ maximum occurs when $\tan\alpha = 2$

(iv) Maximum height



$$h = 20\tan\alpha - 5\sec^2\alpha$$

$$= 20\times 2 - 5(\sqrt{5})^2$$

$$= 15$$

$\therefore$ maximum height is 15m

(v) $x = 20t\cos\alpha$

$$20 = 20t \times \frac{1}{\sqrt{5}}$$

$$1 = \frac{t}{\sqrt{5}}$$

$$t = \sqrt{5}$$

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(20\cos\alpha)^2 + (-10t + 20\sin\alpha)^2}$$

$$= \sqrt{\left(20\times\frac{1}{\sqrt{5}}\right)^2 + \left(-10\sqrt{5} + \frac{40}{\sqrt{5}}\right)^2}$$

$$= \sqrt{80 + 20}$$

$$= 10$$

$\therefore$ The ball hits the wall at a speed of $10\,ms^{-1}$